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Electromagnetic radiation from sinusoidal currents

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Abstract

We study the electromagnetic field radiated by high frequency electric currents in infinite conducting cylinders, with longitudinal or solenoidal flows. We show that the behavior of the radiated fields is very different for radiofrequencies and for higher frequencies. When $\frac{\omega a}{c}$ is large (where a is the cylinder radius), the radiated power varies very rapidly as a function of the frequency and is almost zero for certain values.

1 Current in conducting materials

We consider a homogenous, isotropic, conducting medium, and assume that its permittivity ε , its permeability μ are independent of the frequency. Its conductivity σ is real and is also independent of the frequency if $f \leq 10^{11} Hz$ ⁽¹⁾. In the medium, the Maxwell equations are :

$$\begin{cases} \operatorname{div} \vec{E} = \frac{\rho_f}{\varepsilon} \\ \operatorname{curl} \vec{E} = -\frac{\partial \vec{B}}{\partial t} \end{cases} \quad \begin{cases} \operatorname{div} \vec{B} = 0 \\ \operatorname{curl} \vec{B} = \mu \vec{j} + \varepsilon \mu \frac{\partial \vec{E}}{\partial t} \end{cases} \quad (1)$$

where ρ_f is the charge density and \vec{j} the current density.

The current density is proportional to the electric field $\vec{j} = \sigma \vec{E}$, so the charge conservation $\operatorname{div} \vec{j} + \partial \rho_f / \partial t = 0$ gives $\partial \rho_f / \partial t = -\sigma \rho_f / \varepsilon$, and for a steady current $\rho_f = 0$ and $\operatorname{div} \vec{j} = 0$, everywhere in the conductor.

For a sinusoidal current of frequency $f = \frac{\omega}{2\pi}$, if the complex electric field is $\vec{E}(\vec{r}, t) = \vec{E}(\vec{r}) e^{i\omega t}$ we can write

$$\nabla^2 \vec{E}(\vec{r}) = i\mu\sigma\omega \vec{E}(\vec{r}) - \omega^2\varepsilon\mu \vec{E}(\vec{r}) \quad (2)$$

In the right-hand side of equation (2), the second term (displacement current) is very small compared to the first one (conduction current) even for high frequencies, for instance, in copper $\sigma \sim 5.8 \cdot 10^7 (\Omega m)^{-1}$, the ratio $\varepsilon\omega / \sigma$ is approximately $10^{-18} f$, we can neglect the second term for all frequencies, ranging from 60 Hz to microwaves ($10^{11} Hz$). In a good conductor the current density obeys the law

$$\nabla^2 \vec{j} = i\mu\sigma\omega \vec{j} \quad (3)$$

2 The infinite wire

We first consider an infinite cylindrical wire of radius a , with a sinusoidal volume current circulating in the axis direction, and independent of the z coordinate.

In cylindrical coordinates, $\vec{j} = j(\rho) \exp(i\omega t) \vec{e}_z$ satisfies the equation

$$\frac{d^2 j(\rho)}{d\rho^2} + \frac{1}{\rho} \frac{dj(\rho)}{d\rho} = i\mu\sigma\omega j(\rho) \quad (4)$$

The general solution is

$$j(\rho) = C J_0\left(\frac{1-i}{\delta}\rho\right) + D N_0\left(\frac{1-i}{\delta}\rho\right) \quad (5)$$

J_0 and N_0 are Bessel functions. If the current j is finite at $\rho = 0$, then $D = 0$

$$j(\rho) = C J_0\left(\frac{1-i}{\delta}\rho\right) \quad \text{where } \delta = \sqrt{\frac{2}{\omega\mu\sigma}} \text{ is the skin depth} \quad (6)$$

In copper $\delta \sim \frac{6.6 \cdot 10^{-2}}{\sqrt{f}}$ m, δ varies from 10^{-2} m for a 50 Hz current to 10^{-7} m for microwaves.

C is related to I , the total current amplitude through a section of the wire .

$$C = \frac{I}{2\pi} \frac{1-i}{a\delta} \frac{1}{J_1\left(\frac{1-i}{\delta}a\right)} \quad (7)$$

At any point M , outside the wire, the scalar potential is zero, and the retarded vector potential⁽²⁾ is

$$\vec{A}(r, t) = \frac{\mu_0}{4\pi} \int_{\tau} \frac{\vec{j}(t - \frac{u}{c}, \rho)}{u} d\tau \quad (8)$$

where τ is the wire volume and $u = PM$, P is a point inside the wire with cylindrical coordinates ρ, θ, z , and r is the distance between M and the wire axis.

$$u = \sqrt{z^2 + R^2} = \sqrt{z^2 + r^2 + \rho^2 - 2r\rho \cos \theta} \quad (\text{see Fig.1})$$

As the wire is infinite, the vector potential is independent of the z coordinate of M , M can be placed at $z = 0$

$$\vec{A}(r, t) = \vec{e}_z \frac{\mu_0}{\pi} e^{i\omega t} \int_0^a \rho j(\rho) d\rho \int_0^\pi d\theta \int_0^\infty \frac{e^{-i\omega \frac{u}{c}}}{u} dz \quad (9)$$

The z integral can be evaluated exactly⁽³⁾ and gives $-i\frac{\pi}{2} H_0^{(2)}\left(\frac{\omega}{c}R\right)$ which can be expanded⁽⁴⁾ in Fourier series

$$H_0^{(2)}\left(\frac{\omega}{c}R\right) = J_0\left(\frac{\omega}{c}\rho\right) H_0^{(2)}\left(\frac{\omega}{c}r\right) + 2 \sum_{k=1}^{\infty} J_k\left(\frac{\omega}{c}\rho\right) H_k^{(2)}\left(\frac{\omega}{c}r\right) \cos k\theta \quad (10)$$

The θ integral is zero except for the first term, so

$$\vec{A}(r, t) = -i \vec{e}_z \frac{\mu_0}{2} \pi e^{i\omega t} H_0^{(2)}\left(\frac{\omega}{c}r\right) \int_0^a \rho J_0\left(\frac{\omega}{c}\rho\right) j(\rho) d\rho \quad (11)$$

The ρ integral can also be evaluated exactly⁽⁵⁾

$$\int_0^a \rho J_0\left(\frac{\omega}{c}\rho\right) J_0\left(\frac{1-i}{\delta}\rho\right) d\rho = \frac{\frac{-\omega c}{a} J_1\left(\frac{\omega}{c}a\right) J_0\left(\frac{1-i}{\delta}a\right) + \frac{1-i}{\delta} a J_0\left(\frac{\omega}{c}a\right) J_1\left(\frac{1-i}{\delta}a\right)}{\left(\frac{1-i}{\delta}\right)^2 - \left(\frac{\omega}{c}\right)^2} \quad (12)$$

Since $\left(\frac{\omega\delta}{c}\right)^2 \ll 1$, as have seen in (3), we obtain

$$\vec{A}(r, t) = -i\vec{e}_z \frac{\mu_0}{4} I e^{i\omega t} H_0^{(2)}\left(\frac{\omega}{c}r\right) F(a) \quad (13)$$

$$\text{where } F(a) = J_0\left(\frac{\omega}{c}a\right) - \frac{1+i}{2} \frac{\omega}{c} \delta J_1\left(\frac{\omega}{c}a\right) \frac{J_0\left(\frac{1-i}{\delta}a\right)}{J_1\left(\frac{1-i}{\delta}a\right)}$$

It is easy now to evaluate the fields \vec{E} and \vec{B} and the energy radiated by the wire.

$$\begin{aligned} \vec{E} &= -\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0}{4} I \omega e^{i\omega t} H_0^{(2)}\left(\frac{\omega}{c}r\right) F(a) \vec{e}_z \\ \vec{B} &= \text{curl} \vec{A} = -\frac{\partial A}{\partial r} \vec{e}_\theta = -i \frac{\mu_0}{4} I \frac{\omega}{c} e^{i\omega t} H_1^{(2)}\left(\frac{\omega}{c}r\right) F(a) \vec{e}_\theta \end{aligned} \quad (14)$$

The energy per unit time passing out through a cylinder of unit length is ⁽⁶⁾

$$\mathcal{P} = \frac{\mu_0}{8} I^2 \omega |F(a)|^2 \quad (15)$$

- For radiofrequencies and below, the second term of $F(a)$ is negligible, and as $a \ll \lambda$, $F(a) \simeq J_0\left(\frac{\omega}{c}a\right) \simeq 1$. The result⁽⁷⁾ obtained for infinitesimal wires holds also for wires of finite radius. (see Fig.2)

- For microwaves (see Fig.3), $a \gg \delta$, the asymptotic expansions ⁽⁸⁾ of Bessel functions yield $J_1\left(\frac{1-i}{\delta}a\right) = -iJ_0\left(\frac{1-i}{\delta}a\right)$

$$F(a) = J_0\left(\frac{\omega}{c}a\right) + \frac{1-i}{2} \frac{\omega}{c} \delta J_1\left(\frac{\omega}{c}a\right) \quad (16)$$

When $\omega a/c$ reaches a zero of J_0 , $F(a)$ is reduced to the second term which is very small ($\omega\delta/c \sim 10^{-4}$ for $f = 10^{10}$ Hz), and is strictly zero for a perfect conductor ($\delta = 0$), in this case, the fields are precisely zero everywhere outside the wire, as is shown for surface currents by Abbott and Griffiths⁽⁷⁾.

3 The infinite solenoid

We now assume that the sinusoidal current flows around the axis, in the volume of a conducting medium between two infinite cylinders of radii a and b . (see Fig.4)

We can write $\vec{j} = j_\theta \exp(i\omega t) \vec{e}_\theta$, j_θ depends only on ρ and is a solution of the equation

$$\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dj_\theta(\rho)}{d\rho} \right) - \frac{j_\theta(\rho)}{\rho^2} = \frac{2i}{\delta^2} j_\theta(\rho) \quad (17)$$

The general solution is

$$j_\theta(\rho) = C J_1 \left(\frac{1-i}{\delta} \rho \right) + D N_1 \left(\frac{1-i}{\delta} \rho \right) \quad (18)$$

C and D can be determined with the two requirements :

- The H_z magnetic field component is continuous on either side of the outside surface $H_z(b) = 0$ In the conducting medium $\vec{H} = \frac{i}{\omega\mu} \text{curl} \vec{E}$, the z-component is $H_z = \frac{i}{\omega\mu\sigma} \frac{1}{\rho} \frac{d}{d\rho} (\rho j_\theta)$ and we find

$$C J_0 \left(\frac{1-i}{\delta} b \right) + D N_0 \left(\frac{1-i}{\delta} b \right) = 0 \quad (19)$$

- The total current through a section of length unity (see fig.4) is $I = \int_a^b j_\theta(\rho) d\rho$, then

$$C J_0 \left(\frac{1-i}{\delta} a \right) + D N_0 \left(\frac{1-i}{\delta} a \right) = \frac{1-i}{\delta} I \quad (20)$$

The current flowing in the solenoid becomes

$$j_\theta(\rho) = I \frac{1-i}{\delta} \frac{N_0 \left(\frac{1-i}{\delta} b \right) J_1 \left(\frac{1-i}{\delta} \rho \right) - J_0 \left(\frac{1-i}{\delta} b \right) N_1 \left(\frac{1-i}{\delta} \rho \right)}{\Delta(a, b)} \quad (21)$$

$$\text{with } \Delta(a, b) = N_0 \left(\frac{1-i}{\delta} b \right) J_0 \left(\frac{1-i}{\delta} a \right) - J_0 \left(\frac{1-i}{\delta} b \right) N_0 \left(\frac{1-i}{\delta} a \right)$$

The retarded vector potential due to this current is given by

$$\vec{A}(r, t) = \frac{\mu_0}{4\pi} \int_\tau j_\theta(\rho) \vec{e}_\theta \frac{e^{i\omega(t - \frac{u}{c})}}{u} d\tau \quad (22)$$

Since the geometry is cylindrically symmetric, we may choose M on the x axis, \vec{A} has only a component in the y direction :

$$A_y = \frac{\mu_0}{\pi} e^{i\omega t} \int_a^b \rho j_\theta(\rho) d\rho \int_0^\pi \cos \theta d\theta \int_0^\infty \frac{e^{-i\frac{\omega}{c}u}}{u} dz \quad (23)$$

The z integral is the same as in the preceding case, but now the contribution to the θ integral in the expansion (10) is due to the term $k = 1$

$$A_y(r, t) = -i\mu_0 \frac{\pi}{2} e^{i\omega t} H_1^{(2)} \left(\frac{\omega}{c} r \right) \int_a^b J_1 \left(\frac{\omega}{c} \rho \right) \rho j_\theta(\rho) d\rho \quad (24)$$

The ρ integral is again evaluated exactly ⁽⁵⁾

$$A_y(r, t) = K(r, t) F(a, b) \quad (25)$$

$$\begin{aligned} \text{where } K(r, t) &= i \frac{\pi}{2} \mu_0 I e^{i\omega t} H_1^{(2)} \left(\frac{\omega}{c} r \right) \text{ dependson } r \text{ and } t \\ \text{and } F(a, b) &= a J_1 \left(\frac{\omega}{c} a \right) \\ &\quad - \delta \frac{1+i}{2} \frac{\omega a}{c} \frac{1}{\Delta(a, b)} \left(J_0 \left(\frac{\omega}{c} a \right) X(a, b) - \frac{1+i}{\pi} \frac{\delta}{a} J_0 \left(\frac{\omega}{c} b \right) \right) \end{aligned} \quad (26)$$

is a factor depending only on the system geometry and frequency.

$$X(a, b) = N_0 \left(\frac{1-i}{\delta} b \right) J_1 \left(\frac{1-i}{\delta} a \right) - J_0 \left(\frac{1-i}{\delta} b \right) N_1 \left(\frac{1-i}{\delta} a \right)$$

The expressions for the fields are

$$\begin{aligned} \vec{E} &= -\frac{\pi}{2} \mu_0 I \omega e^{i\omega t} H_1^{(2)} \left(\frac{\omega}{c} r \right) F(a, b) \vec{e}_\theta \\ \vec{B} &= -i \frac{\pi}{2} \mu_0 I \frac{\omega}{r} e^{i\omega t} H_0^{(2)} \left(\frac{\omega}{c} r \right) F(a, b) \vec{e}_z \end{aligned} \quad (27)$$

The time-averaged power dissipated is now

$$\mathcal{P} = \frac{\mu_0}{2} I^2 \pi^2 \omega |F(a, b)|^2 \quad (28)$$

If $b \neq a$, Δ is not zero, since the current is finite everywhere in the solenoid, and we see

immediatly that $F(a, b) = 0$ if $\frac{\omega a}{c}$ is a root of J_1 and $\delta = 0$ (surface currents⁽⁷⁾), whatever is the value of b .

For radii of a few centimeters, $b > a \gg \delta$, and to the first order in $1/z = (1-i)\delta/a$ or $(1-i)\delta/b$, the asymptotic Bessel functions⁽⁸⁾ expansion yields :

$$\begin{aligned} \Delta(a, b) &= \frac{1+i}{\pi} \frac{\delta}{\sqrt{ab}} \sin \left(\frac{1-i}{\delta} (b-a) \right) \\ X(a, b) &= \frac{1+i}{\pi} \frac{\delta}{\sqrt{ab}} \cos \left(\frac{1-i}{\delta} (b-a) \right) \end{aligned} \quad (29)$$

We find easily

$$\begin{aligned} F(a, b) &= a J_1 \left(\frac{\omega}{c} a \right) \\ &\quad - \frac{1+i}{2} \frac{\omega a}{c} \delta \frac{(i \sinh 2x + \sin 2x) J_0 \left(\frac{\omega}{c} a \right) - 2 \sqrt{\frac{b}{a}} (\cosh x \sin x + i \sinh x \cos x) J_0 \left(\frac{\omega}{c} b \right)}{\cosh 2x - \cos 2x} \end{aligned} \quad (30)$$

where $x = \frac{b-a}{\delta}$

For a given value of a , $|F(a, b)|^2$ has the shape shown in Fig.5 for $f = 10^4$ Hz. If $b - a \ll \delta$, the second term in $F(a, b)$ is of the first order in x and

$$F(a, b) = a J_1 \left(\frac{\omega}{c} a \right) + \frac{\omega}{c} a \frac{b-a}{2} J_0 \left(\frac{\omega}{c} a \right) \quad (31)$$

As $b - a$ increases towards 2δ , the radiated power increases slightly, it becomes constant for larger values of b :

$$|F(a, b)|^2 = \left(a J_1\left(\frac{\omega}{c}a\right)\right)^2 + a^2 \frac{\omega}{c} \delta J_1\left(\frac{\omega}{c}a\right) J_0\left(\frac{\omega}{c}a\right) \quad (32)$$

Let us now examine the variation of (31) with frequency (see Fig.6 and Fig.7) $F(a, b)$ is almost zero for $f \leq 10^7$ Hz, then it increases and has a peaked maximum near $\frac{\omega a}{c} = 1.8$ (if $a = 0.1$ m, $f \sim 10^9$ Hz). For higher frequencies, $F(a, b)$ oscillates and is zero when $\frac{\omega a}{c}$ is a root of J_1 .

4 Conclusion

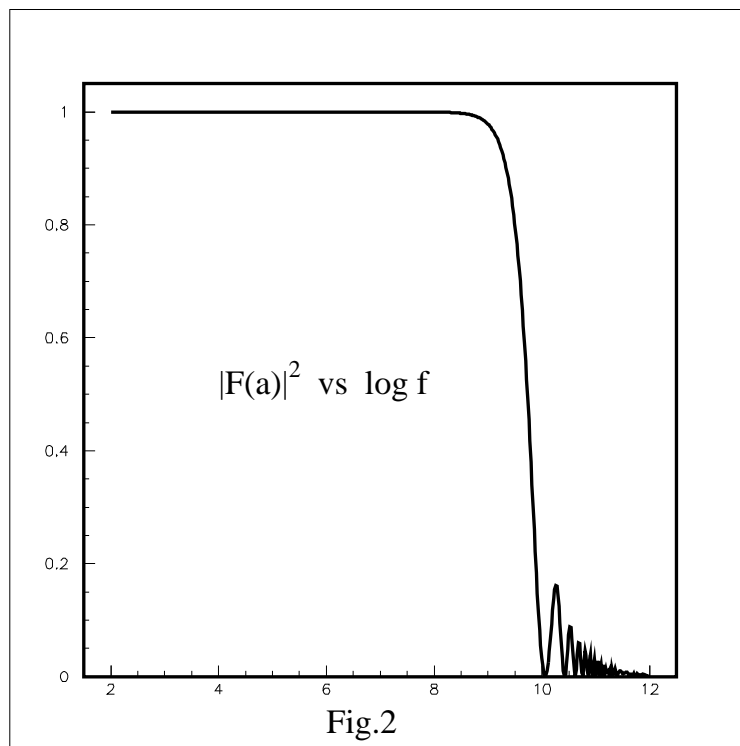
Longitudinal and solenoidal sinusoidal currents flowing in infinite cylinders radiate very differently depending on whether their frequency is below $f_c = \frac{c}{2\pi a}$ or above (where a is the radius of the longitudinal wire and the inner radius of the solenoid).

We see that for a fixed current the power radiated increases as the frequency, Below f_c , the coefficient of $\mu_0 I^2 \omega$ is independent of f , above f_c it oscillates and is practically zero when $\frac{\omega a}{c}$ is a root of J_0 for a longitudinal flow and a root of J_1 for a solenoidal one.

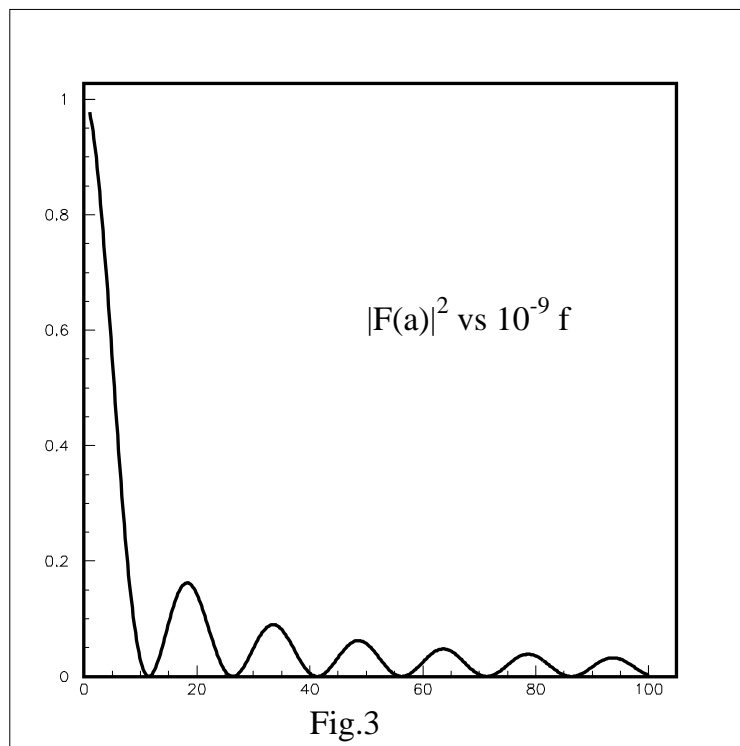
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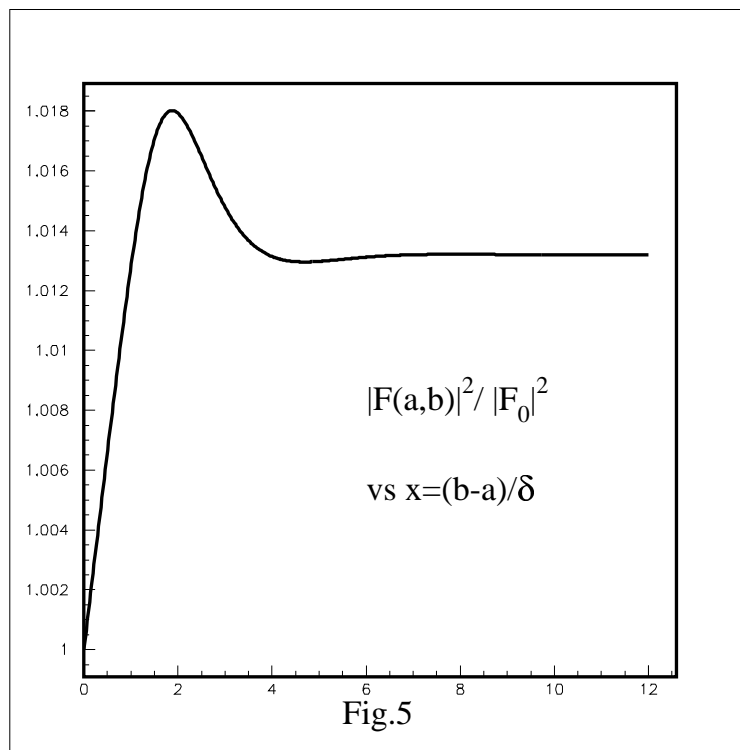
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4. Ref.2, Sec.8.531.2
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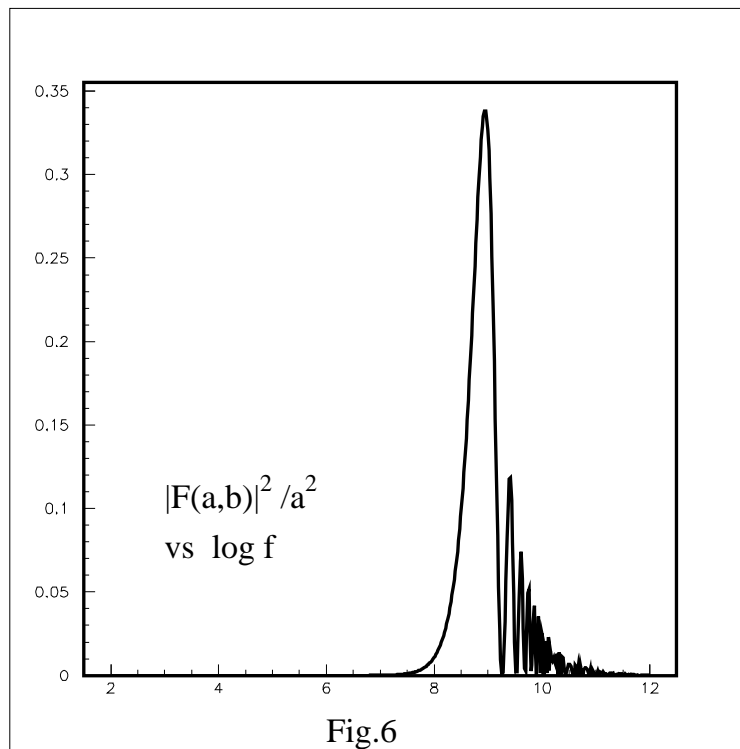
$|F(a)|^2$ for copper, when $a = 0.01$ m and f is varying from 100 to 10^{12} Hz



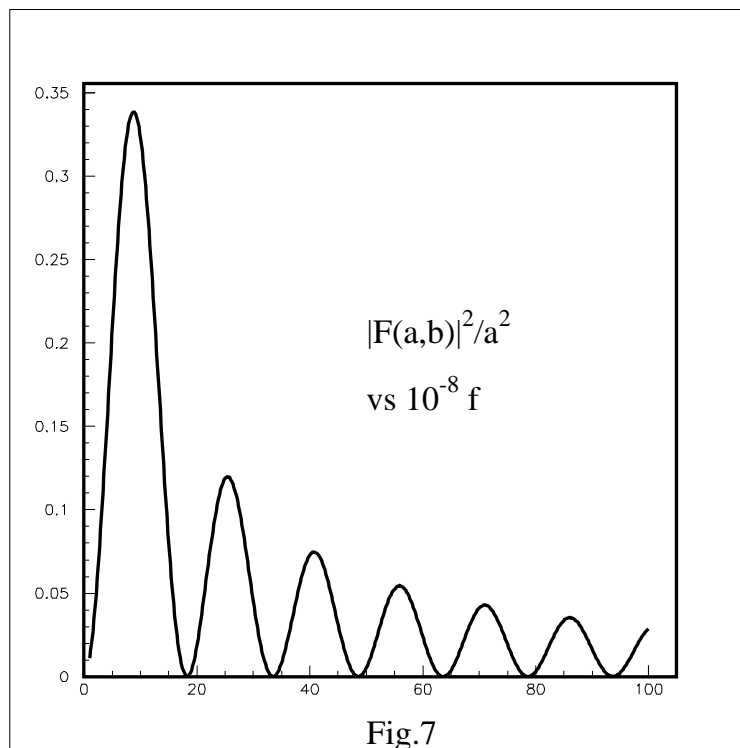
$|F(a)|^2$ for copper, when $a = 0.01$ m and f is varying from 10^9 to 10^{11} Hz



$|F(a,b)|^2/|F_0|^2$ versus x for copper, when $a = 0.1$ m and $f = 10^4$ Hz (F_0 is the value of $F(a,b)$ if $b = a$)



$|F(a,b)|^2/|F_0|^2$, when $a = 0.1$ m and f is varying from 100 to 10^{12} Hz



$|F(a,b)|^2/|F_0|^2$, when $a = 0.1$ m and f is varying from 10^8 to 10^{10} Hz